Tutorial 7 2022.11.16

7.1 Supplementary problems in Assignment 9

Problem 7.1 Let D be the parallelogram formed by the lines $x+y=1, x+y=3, y=2x-3, y=2x+2$. Evaluate the line integral

$$\oint_C dx + 3xydy$$

where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

Problem 7.2 Find a potential for the vector field

$$\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$

in the region obtained by deleting the line $(x,0), x \leq 0$, from $\mathbb{R}^2$.

Problem 7.3 Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field which is defined in $\mathbb{R}^2$ except at the origin. Suppose that it satisfies the component test $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in positive direction, one has

$$\oint_\gamma M dx + N dy = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta, \varepsilon\sin\theta)\cos\theta] d\theta,$$

for all sufficiently small ε. What happens when γ does not enclose the origin?

7.2 Area formula via Green's theorem

Proposition 7.1

Let $D \subset \mathbb{R}^2$ be a domain bounded by the curve γ. Then

$$|D| = \iint_D 1 dxdy = \oint_\gamma xdy = \oint_\gamma -ydx = \oint_\gamma \alpha xdy - (1-\alpha)ydx$$

7.3 Discrete vector calculus

In the following we give a discrete version of vector calculus, the recent topic of this course. The discrete version will help us understand the differential forms better. The content of this tutorial is inspired by Lecture 33 of Math 22a Harvard College.

7.3.1 Discrete forms

Let M be any set (You could pick your favorite finite set). Let $M^k = M \times M \times \cdots \times M$ be the direct product of k copies of M.

Definition 7.1

*A **k-form** α on M is a function $\alpha : M^{k+1} \to \mathbb{R}$ satisfying*

$$\alpha(u_1, \ldots, u_i, u_{i+1}, \cdots, u_{k+1}) = -\alpha(u_1, \ldots, u_{i+1}, u_i, \cdots, u_{k+1}).$$

for $u_i \in M, i=1,\cdots,k+1$ *The space of* k*-form on* M *is denoted by* $\Omega^k(M)$ ♣

We have

$$\alpha\left(u_1,\cdots,u_{k+1}\right)=0 \quad \text{if} \quad u_i=u_j$$

and

$$\alpha\left(u_1,\ldots,u_i,\cdots,u_j,\cdots,u_{k+1}\right)=-\alpha\left(u_1,\ldots,u_j,\cdots,u_i,\cdots,u_{k+1}\right).$$

Example 7.1

1. A zero-form is just a function on M.
2. A one-form α is a function over $M\times M\to\mathbb{R}$ satisfying $\alpha(u,v)=-\alpha(v,u)$. This is the analogy of vector fields.

From a k-form α we could construct a $k+1$-form.

Definition 7.2

The ***exterior derivative*** $d:\Omega^k(M)\to\Omega^{k+1}(M)$ *is the map defined as*

$$\begin{aligned}
& d\alpha_k\left(u_1,\ldots,u_{k+2}\right)\\
&=\alpha_k\left(u_1,\cdots,u_{k+1}\right)-\alpha_k\left(u_1,\cdots,u_k,u_{k+2}\right)\\
&+\alpha_k\left(u_1,\cdots,u_{k-1},u_{k+1},u_{k+2}\right)+\cdots\\
&=\sum_{i=1}^{k+2}(-1)^{k+2-i}\alpha_k\left(u_1,\cdots,u_{i-1},\hat{u}_i,u_{i+1},\cdots,u_{k+2}\right)
\end{aligned}$$

where $\alpha_k\in\Omega^k(M)$ *is a* k*-form and* $u_1,u_2,\cdots,u_{k+2}\in M$. *By a hat* $\hat{u}_i$ *on* u_i *we exclude the i-th argument* u_i. ♣

Example 7.2

1. Let α_0 be a zero-form, so it is a function on M. Then

$$\begin{aligned}
d\alpha_0(u,v)&=\alpha_0(u)-\alpha_0(v)\\
d:\Omega^0(v)&\to\Omega^1(v)
\end{aligned}$$

$$\text{Function} \xrightarrow{gradient} \text{Vector field.}$$

2. Let α_1 be a one-form, so it is a function on $M\times M$, then

$$\begin{aligned}
d\alpha_1(u,v,w)&=\alpha_1(u,v)-\alpha_1(u,w)+\alpha_2(v,w)\\
d:\Omega^1(v)&\to\Omega^2(v)
\end{aligned}$$

$$\text{Vector field} \xrightarrow{curl} \text{Vector field.}$$

3. Likewise, the differential of a 2-form is an analogy of taking divergence of a vector field if valued on a tetrahedron (i.e. 4 points).

Proposition 7.2 (Component test)

$d^2=0$. ♠

Proof Let α_0 be a zero form. Then $d(d\alpha_0)(u,v,w)=d\alpha_0(u,v)-d\alpha_0(u,w)+d\alpha_0(v,w)=(\alpha_0(u)-\alpha_0(v))-(\alpha_0(u)-\alpha_0(w))+(\alpha_0(u)-\alpha_0(w))$.

And you can work out the proof for general cases for any k-form.

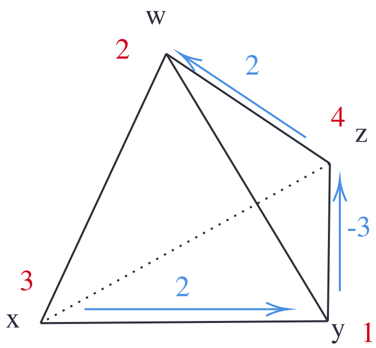

Figure 7.1: $M = \{x, y, z, w\}$

Remark This is the analogy of the following

1. The curl of the gradient of any scalar field φ is always the zero vector field
$$\nabla \times (\nabla \varphi) = \mathbf{0}$$
2. The divergence of the curl of any vector field (in three dimensions) is equal to zero:
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Pictorially, we could view zero-forms as functions on vertices, one-forms as functions on directed edges, 2-forms as functions on directed triangle faces, 3-forms as functions on directed tetrahedrons, and k-forms as functions on directed k-dimensional simplex. As in figure 7.1, the form on a tetrahedron serves as an analogy of the vector calculus in $\mathbb{R}^3$.

7.3.2 Line integrals and Poincaré's Lemma

Definition 7.3

*A **curve** of M is an element of $\coprod_{k=1}^{\infty} M^k$. If $c \in \coprod_{k=1}^{\infty} M^k$ then we can write it as $c = (u_1, u_2, \cdots, u_m)$ for $u_i \in M, i = 1, 2, \cdots, m$. The curve is **regular** if $\forall i, j, u_i \neq u_j$.* ♣

Definition 7.4

*Let α_1 be a one-form on M, define the **line integration** of α_1 along a curve c as*
$$\int_c \alpha_1 := \sum_{i=1}^{m-1} \alpha(u_i, u_{i+1})$$
♣

Definition 7.5

*A one-form α_1 is called **conservative** if there is a zero-form α_0 such that $\alpha_1 = d\alpha_0$. In this case, α_0 is called the **potential** of α_1.* ♣

Example 7.3 In the figure 7.1 a function f is defined on $M = \{x, y, z, w\}$ as $f(x) = 3, f(y) = 1, f(z) = 4, f(w) = 2$. Then the integration of df along $c = (x, y, z, w)$ is
$$\int_c df = df(x,y) + df(y,z) + df(z,w) = f(x) - f(y) + f(y) - f(z) + f(z) - f(w) = f(x) - f(w) = 1$$

Theorem 7.1 (Fundamental theorem of Line integral)

Let $\alpha_1 = d\alpha_0$ be a conservative one-form and $c = (u_1, u_2, \cdots, u_m)$ be a curve of M, then

$$\int_C \alpha_1 = \alpha_0(u_1) - \alpha_0(u_m)$$

♡

Proof Simple exercise left for the readers.

Therefore, the integration of a conservative 1-form does not depend on the choice of the curves connecting two fixed points.

Proposition 7.3

A one-form α_1 is conserved $\iff d\alpha_1 = 0$

♠

Proof By proposition 7.2, we just need to prove that $d\alpha_1 = 0$ implies $\alpha_1 = d\alpha_0$ for some zero form α_0. In fact, let u be any point in M, and define $\alpha_0(v) := \alpha_1(v, u)$. Then

$$d\alpha_0(u_1, u_2) = \alpha_0(u_1) - \alpha_0(u_2) = \alpha_1(u_1, u) - \alpha_1(u_2, u)$$

Since $d\alpha_1 = 0$, we have $d\alpha_1(u_1, u_2, u) = \alpha_1(u_1, u_2) - \alpha_1(u_1, u) + \alpha_1(u_2, u)$. Therefore,

$$d\alpha_0(u_1, u_2) = \alpha_1(u_1, u) - \alpha_1(u_2, u) = \alpha_1(u_1, u_2)$$

Proposition 7.3 is not a special phenomenon only for 1-form.

Definition 7.6

A k-form α_k is ***closed*** *if $d\alpha_k = 0$.*

A k-form α_k is ***exact*** *if $\alpha_k = d\alpha_{k-1}$ for some $k-1$-form α_{k-1}.*

♣

Theorem 7.2 (Poincaré's Lemma)

A k-form α_k is closed if and only if it is exact.

♡

Proof By proposition 7.2, we just need to show that a closed k-form α_k is exact. Since α_k is closed, it satisfies

$$\begin{aligned} & d\alpha_k\left(u_1, \ldots, u_{k+2}\right) \\ & = \sum_{i=1}^{k+2}(-1)^{k+2-i}\alpha_k\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+2}\right) = 0 \end{aligned}$$

For any $u \in M$, we define $\alpha_{k-1}(u_1, u_2, \cdots, u_k) := \alpha_k(u_1, u_2, \cdots, u_k, u)$. Then

$$\begin{aligned} & d\alpha_{k-1}\left(u_1, u_2, \cdots, u_{k+1}\right) \\ & = \sum_{i=1}^{k+1}(-1)^{k+1-i}\alpha_{k-1}\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+1}\right) \\ & = \sum_{i=1}^{k+1}(-1)^{k+1-i}\alpha_k\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+1}, u\right) \\ & = \alpha_k\left(u_1, \ldots, u_{k+1}\right) \end{aligned}$$

7.3.3 Integration on domains and Stoke's theorem

Mimicking the definition of line integral, we could define the integral of an arbitrary k-form α_k.

Definition 7.7

*A **signed simple domain** D is an ordered tuple of points in M with a sign $+$ or $-$, so we could write it as $D = \pm(u_1, \cdots, u_m), u_i \in M$. We call $m-1$ the **dimension** of D. We define $-D$ to be $\mp(u_1, \cdots, u_m)$. A **domain** V is a set consisting of finite many signed simple domains $\{D_1, D_2, \cdots, D_l\}$, we also write $V = D_1 \vee D_2 \vee \cdots \vee D_l$. Here we allow repetitions, i.e., it may happen that D_i and D_j represent the same signed simple domain. We say V is **pure of dimension** m if D_i is of dimension m for all $i = 1, \cdots, l$.* ♣

Definition 7.8

If $D = (u_1, \cdots, u_m)$ is a signed simple domain, let $F_i = (-1)^{m-i}(u_1, \cdots, \hat{u}_i, \cdots, u_m)$. The signed simple domains F_i are called the facet of D.
*The **boundary** ∂D of a D is defined as*

$$\partial D := F_1 \vee F_2 \vee \cdots \vee F_m.$$

*The **boundary** of a domain V is the disjoint union of the boundary of the signed simple domains in V. By 'disjoint' it means still we allow repetitions, i.e., if a signed simple domain D lies in the boundaries of both D_1 and D_2 of V, then we list D twice in ∂V.* ♣

Definition 7.9

*The **integration** of a k-form α_k over a signed simple domain $D = \pm(u_1, \cdots, u_m)$ is*

$$\int_D \alpha_k = \pm \sum_{1 \leqslant s_1 < s_2 < \cdots < s_{k+1} \leqslant m} \alpha_k\left(u_{s_1}, \ldots, u_{s_{k+1}}\right)$$

*The **integration** of a k-form α_k over a domain $V = D_1 \vee D_2 \vee \cdots \vee D_l$ is*

$$\int_V \alpha_k = \sum_{i=1}^{l} \int_{D_i} \alpha_k$$

♣

Example 7.4 The notation for a curve $c = (u_1, \cdots, u_m)$ of M in definition 7.3 could also be written as $c = (u_1, u_2) \vee (u_2, u_3) \vee \cdots \vee (u_{m-1}, u_m)$, and the line integration for a one-form α_1 over c is the same as the integration of α_1 over the domain $(u_1, u_2) \vee (u_2, u_3) \vee \cdots \vee (u_{m-1}, u_m)$.

Theorem 7.3 (Stokes' theorem)

Let α_k be a k-form and V be a domain which is pure of dimension $k+1$, then

$$\int_V d\alpha_k = \int_{\partial V} \alpha_k$$

♡

Proof We may assume $V = D = (u_1, \cdots, u_{k+2})$ is a signed simple domain of dimension $k+1$.

According to the definition

$$
\begin{aligned}
\int_D d\alpha_k &= \sum_{1\leqslant s_1<s_2<\cdots<s_{k+2}\leqslant k+2} d\alpha_k\left(u_{s_1},\ldots,u_{s_{k+2}}\right) \\
&= d\alpha_k(u_1,\cdots,u_{k+2}) \\
&= \sum_{i=1}^{k+2}(-1)^{k+2-i}\alpha_k\left(u_1,\cdots,u_{i-1},\hat{u}_i,u_{i+1},\cdots,u_{k+2}\right) \\
&= \sum_{i=1}^{k+2}(-1)^{k+2-i}\int_{(u_1,\cdots,u_{i-1},\hat{u}_i,u_{i+1},\cdots,u_{k+2})}\alpha_k \\
&= \sum_{i=1}^{k+2}\int_{F_i}\alpha_k \\
&= \int_{F_1\vee\cdots\vee F_{k+2}}\alpha_k \\
&= \int_{\partial D}\alpha_k
\end{aligned}
$$